CAS as Pedagogical Tools for Teaching and Learning Mathematics

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Abstract: Computer algebra systems (CAS) make us think about WHAT and HOW we teach mathematics. In this article we focus on the HOW by developing a two level framework for understanding, categorizing, and planning the use of technology, in particular CAS, in teaching and learning mathematics. First we look at two kinds of support a tool can give: *automation* and *compensation*. Second we look at four pedagogical approaches for teaching and learning mathematics which are greatly facilitated when using CAS properly: *trivialization*, *experimentation*, *visualization*, and *concentration*. Based on this framework we introduce the *scaffolding method* as a pedagogically justified sequence of using and not using technology to achieve certain teaching goals.

(0) Introduction

In mathematics education, computer algebra systems (CAS) have sparked off a worldwide discussion about the value and sensefulness of what we teach in a traditional math class. CAS point us to the fact that we focus too much on the teaching of the craftsmanship of performing operations, i.e. skills which a computer can so easily replace. Hence CAS make us think about WHAT we teach.

There are thousands of ways of using CAS for teaching – good ones and bad ones. Bad or, better, improper approaches often come from technology freaks among the teachers, who use the CAS just because it exists and because it is possible to use it in a certain way. But CAS should never drive the math we teach – our math (teaching goals) should drive the (use of) CAS! I fully agree with Helmut Heugl, the director of the Austrian Derive and TI-89/92 Projects (which involved almost 6,000 students), who said: *"If it is not pedagogically justified to use CAS, it is pedagogically justified not to use CAS."* So: CAS also make us think about HOW we teach.

The WHAT is discussed in [Kutzler 2001], where we attempt to give an answer. In this article we focus on the HOW. We show the different roles CAS can play as tools for teaching and learning mathematics. Our approach is strictly driven by math teaching goals, i.e. we advocate the importance of a pedagogically justified use of CAS. A lot of the following thoughts are applicable also for other tools such as graphing calculators.

(1) Tools I: Automatization and Compensation

In a speech Albert Einstein gave to educators in 1936 he gave the following remarkable analogy: *"If a young man has trained his muscles and physical endurance by gymnastics and walking, he will later be fitted for every physical work. This is also analogous to the training of the mind and the exercising of the mental and manual skill."* [Einstein 1956]

Following this analogy I start with looking at the two disciplines *mathematics* (as an intellectual achievement) and *moving&transportation* (as a physical achievement) and compare the roles of technology therein. This comparison was also stimulated by Frank Demana.

The most elementary method of moving is *walking.* Walking is a physical achievement obtained with mere muscle power. The corresponding activity in mathematics is *mental calculation* (mental arithmetic and mental algebra.) Mental calculation requires nothing but „brain power".

Riding a bicycle is a method of moving, where we employ a mechanical device for making more effective use of our muscle power. Compared to walking we can move greater distances or we can move faster. The corresponding activity in mathematics is *paper and pencil calculation*. We use paper and pencil as „external memory" which allows us to use our brain power more efficiently.

Another method of moving is *driving a car*. The car is a device that produces movement. The driver needs (almost) no muscle power for driving, but definitely needs new skills: (S)he must be able to start the engine, to accelerate, to steer, to brake, to stick to the traffic regulations, etc. The corresponding activity in mathematics is *calculator/computer calculation*. The calculator or computer produces the result, while its user needs to know how to operate it and what button to push in a certain situation.

What method of moving is sensible in which situation? If we ask somebody to obtain today's newspaper from a 150 meter distant newsstand, the person probably will walk. In case the newsstand is 1,000 meters away, a bicycle may be the most reasonable means of transportation. In case the distance to the shop is 10 miles, it is advisable to use a car, in particular if there is only half an hour for bringing the newspaper. In mathematics, the sensible use of technology is accordingly: The multiplication of two one-digit numbers is best done mentally. Two two-digit numbers can well be multiplied using paper and pencil, while for the product of two seven-digit numbers one will want to use a calculator.

One could throw in that *„many students use a calculator to obtain the product of 7 and 9"*, hence they are likely to lose the skill of performing mental arithmetic. This is a clear case of improper use of technology, which is not specific to mathematics education but happens in all areas. Some people misuse their car by driving 150 meters to the next newsstand. Those, who do so, harm themselves (lack of physical exercise) and our environment (through the exhaust fumes). Despite the possible misuse of the car we do not demand its abolition. Similarly we should not banish calculators and computers just because some students might use them improperly. As much as we needed (and still need) to create a general awareness that physical exercises are essential for physical fitness and health, we need to create the awareness that mental exercises (such as mental arithmetic and mental algebra) are essential for mental wellness and fitness. Nevertheless it goes without saying that it is not enough to try to persuade the students to do simple computations manually. But there is a rather simple method we can use to enforce this in an elegant manner. We simply have to split the exams into two parts: A compulsory part and a voluntary (freestyle) part. This is analogous to ice skating, where the sports person gets two independent scores which are added to give the total score. In the compulsory part of a math exam we would primarily test mental skills, allowing no technology – not even a four-function calculator. In the freestyle part we would focus on testing problem solving skills and students would be allowed to use any kind of technology, in particular CAS. More on this in [Kutzler 2000] (or [Kutzler 2001]) and [Herget etal 2000].

The analogy is not finished yet. What, if the person, who we asked to get a newspaper from a 150 meter away newsstand, can't walk properly because (s)he is physically challenged or has a broken leg? For her or him, walking 150 meters may be very difficult, if not impossible. There is technology available to help such people, for example a wheel chair. *Using a wheel-chair* is a method of technology-supported moving, where technology compensates for a physical weakness. But weaknesses occur also with mental activities. I give an example from mathematics teaching: A student with a weakness in solving systems of linear equations will find it very difficult if not impossible to solve analytic geometry problems, simply because the solving of a system of equations is a subproblem encountered very frequently in analytic geometry. It is not only an act of humanity, but our pedagogical duty to provide a "mathematically challenged" student with a tool which compensates his or her weakness (i.e. let the student solve systems of equations by pushing a button). Only then will the student have a fair chance to do analytic geometry despite the weakness. This is a serious equity issue!

Moving&Transportation (physical)	*Mathematics (intellectual)*
walking	mental calculation
riding a bicycle	paper & pencil calculation
driving a car	calculator/computer calculation (automation)
using a wheel chair	calculator/computer calculation (compensation)

As we will demonstrate later with more examples, calculators and computers can be excellent mathematical compensation tools which allow mathematically challenged students to deal with advanced topics. It goes without saying, that the ultimate goal in mathematics teaching is to weed out all weaknesses in skills that are regarded essential. (Needless to say that CAS also make us reconsider what skills we consider essential, see [Herget etal 2000] for a discussion and proposal.) A physically challenged person need not be tied to a wheel chair for the rest of her or his life. A physician will endeavor to repair a patient's physical challenge as much as possible with an individual therapy. Similarly, a teacher should endeavor to repair a student's mathematical challenge with an appropriate individual therapy. In both cases we will facilitate the "patient's daily life with their challenge" – i.e. the time outside the therapy, be it shopping or analytic geometry – by providing an appropriate compensation tool such as a wheel chair or a calculator/computer.

Automation and *compensation* are the two most elementary types of support any tool can give. These exist on a general level and are not specific to teaching and learning. But, needless to say, they certainly also apply for using CAS tools to teach and learn mathematics. As a tendency, CAS are more likely to serve as an automation tool for what we today call "good" students and CAS are more likely to serve as a compensation tool for "mathematically challenged" students. I write these words in quotes because they are relative to our teaching. When we teach mathematics differently in terms of contents and style, the split into "good" and "challenged" students most probably will be different.

Teachers are supposed to guide students in their learning, so they do have the pedagogical duty to use whatever is available to facilitate the learning process. Using compensation tools is the more urgent aspect, because this is how we can help the mathematically challenged students. For them, a change is much more important than for those students who already do well.

(2) Tools II: Trivialization, Experimentation, Visualization, and Concentration

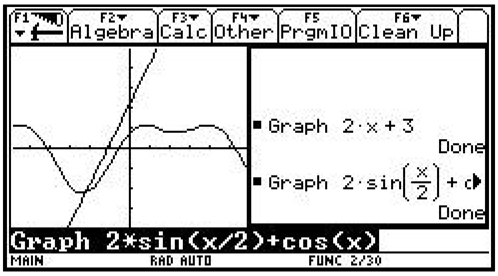
In this section we get specific now regarding the role of technology in teaching and learning. In the sequel we demonstrate and discuss how one can use technology, in particular CAS, as pedagogical tools by looking at four powerful approaches in mathematics teaching and learning: trivialization, experimentation, visualization, and concentration.

(2.1) Trivialization

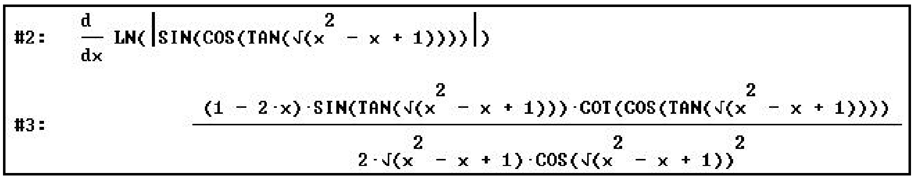
The car broadened our „moving and transportation horizons" by trivializing moving up to certain distances. Similarly, the calculator broadens our „calculation horizons."

Remember the „old days" before scientific calculators when exam questions or homework problems had to be chosen very carefully so that all intermediate and final results were „nice". A „nice" result was an integer, a simple fraction, or a simple radical which, later in the calculation, often would disappear again. This was important, as otherwise the students would have had to use (spoil?) most of their time performing arithmetic operations. With a scientific calculator one can multiply two seven-digit numbers as quickly as two one-digit numbers. Hence the scientific calculator trivializes the performing of arithmetic operations.

Drawing the graph of a linear function (e.g. $y=2x+3$) is simple once you know the geometric meaning of the two coefficients. Only a glimpse of talent, a pencil, and a straightedge are enough to produce a proper graph. Drawing the graph of a function such as $y=2\sin(x/2)+\cos(x)$ is much more difficult and the production of a proper graph requires a reasonable degree of talent for drawing. With a graphics calculator one can plot both functions within the same amount of time and talent. The graphics calculator trivializes the production of graphs. (Following is a respective screen image of a TI-92.)



Computing the first derivative of $y=x^2$ is simple once you know the differentiation rule for powers. Determining the first derivative of $y=\ln(\left|\sin(\cos(\tan(\sqrt{x^2-x+1})))\right|)$, however, is a lot of work even for a good mathematician. The algebraic calculator can manage both examples within seconds. The algebraic calculator trivializes algebraic (symbolic) computations. (Following is a respective screen image of the PC program Derive 5.)



A landmark paper about trivialization of algebraic computation in mathematics teaching is [Buchberger 1989], where the WhiteBox/BlackBox principle was introduced.

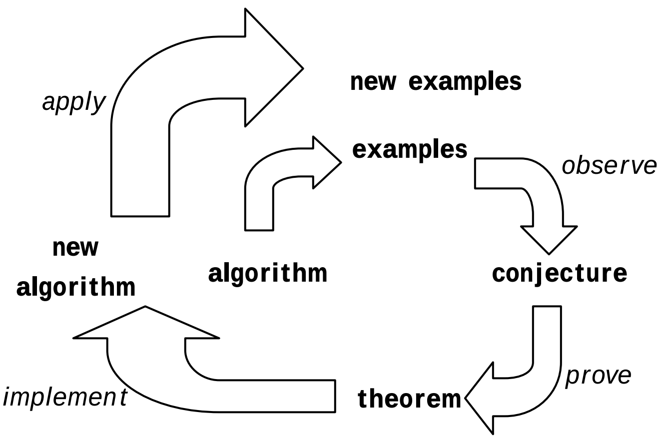
Moving and transportation tasks that have been considered difficult in earlier times nowadays are fulfilled routinely with cars or other transportation devices. Similarly CAS used in teaching mean that we can tackle

- (more) complex problems and
- (more) realistic problems

This is *one* aspect of using CAS, in my opinion the least important.

(2.2) Experimentation

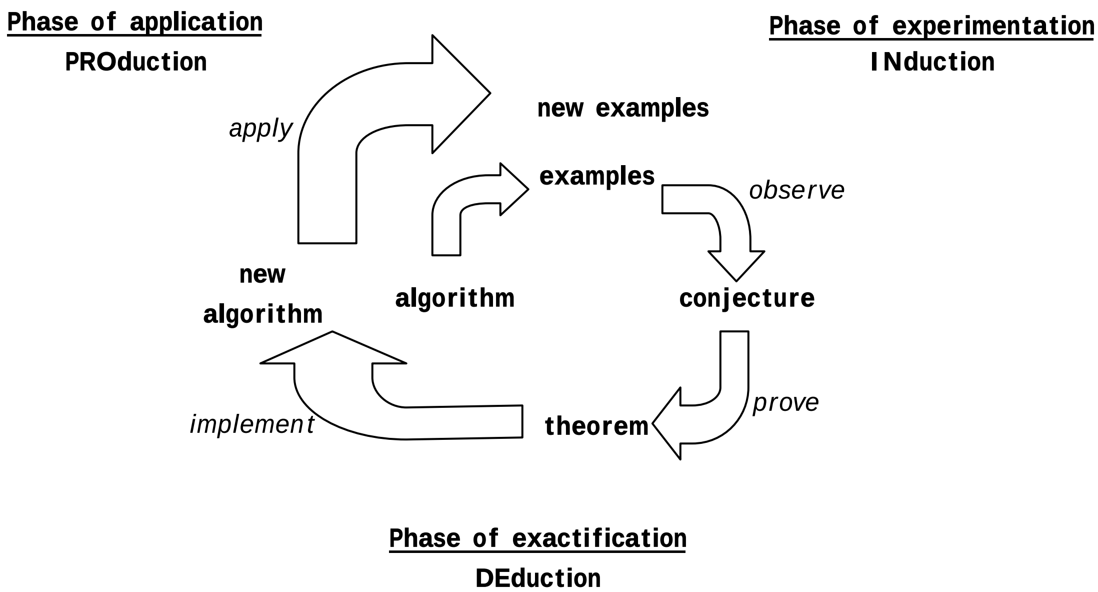
How did we discover all the mathematics we know today and how do we find even more mathematics? According to one of the epistemologically oriented theories one can visualize the main steps of these discoveries as follows: Applying known algorithms produces *examples*. From the examples we *observe* properties, which are inductively expressed as a *conjecture*. Proving the conjecture yields a *theorem*, i.e. guaranteed knowledge. The theorem‘s algorithmically usable parts are *implemented* in a *new algorithm.* Then the algorithms are *applied* to new data, yielding *new examples*, which lead to new observations, new conjectures, ...



This picture of a spiral, which demonstrates the path of discovery of (mathematical) knowledge, was proposed by Bruno Buchberger. A detailed description of *Buchberger's Creativity Spiral* and references to related models can be found in [Heugl/Klinger/Lechner 1996]. While for the purpose of this article we interpret Buchberger's Creativity Spiral globally and generally as a process of developing mathematical knowledge, it can also be seen as a visualization of the (inductive) concept development process according to the constructivist approach of Piaget ([Piaget 1972], [Marin/Benarroch 1994]).

In this spiral we find three phases. During the *phase of experimentation* one applies known algorithms to generate examples, then obtains a conjecture through observation. During the *phase of exactification* the conjecture is turned into a theorem through the method of proving, then algorithmically useful knowledge is implemented as a new algorithm. During the *phase of application* one applies algorithms to real or fictitious data. Typically, the solution of real problems serves the purpose of mastering or facilitating life, while the solution of fictitious problems serves the purpose of entertainment/diversion (e.g. mind puzzles) or the finding of new knowledge (i.e. the satisfaction of scientific curiosity).

These three phases can more elegantly be denoted simply as INduction, DEduction, and PROduction. This is possible for both interpretations of Buchberger's Creativity Spiral. (Although Piaget's concept development process is inductive as a whole, it does contain inductive, deductive, and productive subprocesses.)



In its beginnings mathematics was an experimental science, i.e. it consisted only of the phases of experimentation and application. Then the Greek applied to it the deductive methods of their philosophy (i.e. they added the phase of exactification), thus establishing mathematics as the deductive science as we know it today. "Recently" (in terms of history) a group around the French mathematician Dieudonne (known under the name „Bourbaki") restructured the mathematical knowledge using the system of „definition-theorem-proof-corollary-...". This Bourbaki system, being developed for the purpose of inner-mathematical communication, comprises only the phases of exactification and application and has become characteristic to modern mathematics. But then Bourbakism gradually lodged itself in teaching and learning. It has become customary to teach mathematics by deductively presenting mathematical knowledge, then asking the students to learn it and apply it to solve problems. But this is highly unnatural! Most of today's psychological theories of learning consider learning to be an inductive process in which experimentation plays a key role. This is why Freudenthal demanded that *"we should not teach students something that they could discover themselves"* (see [Freudenthal 1979]).

Hardly any mathematician on this planet could do mathematical research the way we demand our students to get into this subject. But: A student has to „locally“ build his individual little „house of mathematics“ while a scientist does pretty much the same „globally“, i.e. on a much larger scale. For both the scientist and the student a substantial part of knowledge acquisition happens during the phase of experimentation. From this point of view it becomes understandable why so many students are at loggerheads with mathematics, and one will demand that experimentation obtains its due position within the teaching of mathematics. Phases of experimentation should complete the traditional teaching methods – not substitute them! This is not a plea for returning to Egyptian experimental mathematics but is a plea for mathematics teaching going through all three phases of the above spiral.

However, it is understandable that, within the framework of today‘s curricula, there was hardly any experimentation in our classrooms. Experimentation, performed with paper and pencil, is both timeconsuming and errorprone. Within the time available at school, students could produce only a very small number of examples for the purpose of observing and discovering, and a hefty portion of these examples probably would be faulty due to calculation errors. There is nothing you can observe from only a few, partly wrong examples! Look at a typical example from geometry. Say, we want to teach our students that in every triangle the three altitudes intersect in one point. We might ask them to draw five triangles and construct the three altitudes in each. What happens? Most of our students – being lousy draftsmen – will find that in three or four of their five triangles the altitudes do NOT intersect in one point. And this should convince them that this is a true geometric theorem ...?!

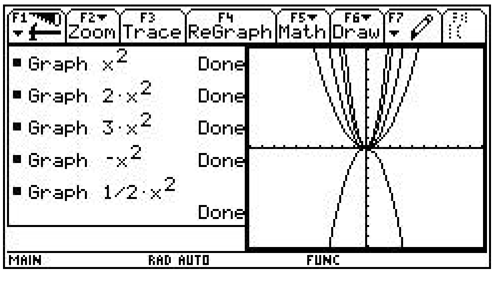
From now on CAS and similar tools (such as Dynamic Geometry Software) enable students to experiment within almost all topics treated in mathematics teaching. Thestudents can do large numbers of examples in a short time and the electronic assistant guarantees the properness of the results. Talking about an assistant: Historic records indicate that great mathematicians such as Carl Friedrich Gauss employed herds of human „calculators“ without which they would not have made most of their famous findings.

The great genius Johann Wolfgang Goethe called for „learning through doing and observing“ which is exactly the “phase of experimentation” in the above picture. Using CAS we can now meet Goethe’s demand.

(2.3) Visualization

Visualization means illustration of an object, fact, or process. The result can be graphic, numeric, or algebraic. Today, the term is mostly used for graphic illustrations of algebraic or numeric objects or facts. (The term is used either for the process of illustration or for the result of the illustration process.) Visualization as a technique of teaching mathematics has become important in those countries, where graphics calculators are widely used.

Today it is mainly used to acquire the competence of changing between representations, mostly for the purpose of studying the correspondance between algebraic and graphic representations. A typical example is studying how the parameter a effects the shape of the graph of the function $y = a \cdot x^2$:



If we ask our students to manually draw the graphs of x^2, $2x^2$, $3x^2$, $\frac{1}{2}x^2$, and $-x^2$ the “good” students are doing fine, but for most of the mathematically challenged students this turns into a laborsome and tedious graph drawing exercise, during which they completely forget WHY they had to draw these graphs in the first place. Hence the teaching goal got lost because of their poor drawing skills. Using CAS or graphing calculators will help these students to stay connected with the goal of finding the relationship between the value of a and the shape of the graph. Frank Demana and Bert Waits are the leading advocates of a teaching style that is called the „power of visualization“ (see [Demana/Waits 1990, 1992, 1994]).

In the psychology of learning scientists discovered the concept of *reinforcement* and showed, that reinforcement works best if it follows the action immediately. An example from everyday life is a child who puts its hand on a hot stove. The immediate pain is the best prerequisite for the child to learn not to do this anymore. If the pain would be felt several minutes later, the child probably would not connect it with the (long ago) touching of the hotplate and it probably would not learn anything.

When using a calculator as a visualization tool, the immediate feedback is of central importance. If you enter into your calculator x^2, then press the proper key, the corresponding graph appears only fractions of a second later. The resulting picture can be discussed, the graph can be associated with the expression, etc. Consider a mathematically challenged student and compare the learning effect when the student has to draw the graph manually with when the student is allowed to use a calculator: A manually produced graph would take too much time and most probably it would have only a vague resemblance with the true graph. („What can you observe from wrong examples?“) Only with the help of the calculator this student has a realistic chance to discover and memorize the correspondance between the expression and the (proper!) graph. Producing graphs with paper and pencil certainly continues to be a worthwhile activity which is important in learning to *understand* the correspondance between algebraic and graphic representations. However, immediacy and correctness are such crucial psychological factors, that providing mathematically challenged students with an appropriate tool (such as a CAS or a graphing calculator) becomes a pedagogical duty.

This example very clearly demonstrates: If and how a teacher uses a tool for supporting an activity depends on the pedagogical goals connected with the activity. This also means, in particular, that the teacher becomes more and more important in a technology-supported mathematics education, hence the importance of teacher preservice and inservice training grows. While the power of visualization is widely used with graphing calculators, it is also an important approach with algebraic data as will be demonstrated with the example in the next section.

(2.4) Concentration

We can compare teaching and learning mathematics with building a house, the „house of mathematics“. The topics which we teach and the dependencies between them are comparable to the storeys of a house. Before one can build the house’s second storey, one has to complete its first. Similarly, the treatment of almost any mathematical topic requires the mastery of earlier learned topics. We demonstrate this using the topic „solving of a linear equation in one variable“.

We look at the equation $5x\text{-}6=2x+15$. One has to transform it into the form $x=\ldots$. This is achieved through choosing and applying an appropriate sequence of equivalence transformations. Typically, the student is advised to „bring terms with x to one side of the equation“ and to „bring all other terms to the other side“. Therefore we start by subtracting $2x$:

$$5x-6 = 2x+15 \;\mid\; -2x$$

After *choosing* this equivalence transformation, we *apply* it to both sides of the equation i.e. we have to simplify:

$$3x-6 = 15$$

Now we have to choose another equivalence transformation, for example +6:

$$3x-6 = 15 \qquad \mid +6$$

And we simplify again:

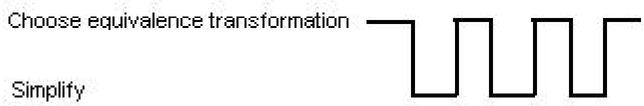
$3x = 21$

We are interested in the *practice* of teaching mathematics. In particular we want to know why students make what errors. A typical mistake at this stage starts with the following argument: *„There is a 3 in front of the variable x. To get rid of the 3 I need to subtract 3"*. This student is most likely to write ...

$3x = 21 \qquad | -3$
$x = 18$

..., believing that the equation is solved.

What goes wrong and how can technology help to make it better? An analysis of the steps taken above reveals two alternating tasks: (1) the choice of an equivalence transformation and (2) the simplification of algebraic expressions. Here, the choice of an equivalence transformation is a higher-level task insofar as it is the essence of the strategy for finding the solution of an equation. It is the new skill which the student has to learn when learning to solve equations. The simplification of expressions is a lower-level task, for which the teacher has to assume that the student is sufficiently well trained.



This picture demonstrates that a student, while trying to learn a new skill, repeatedly has to interrupt the learning process in order to perform a calculation. This is as if one would repeatedly be interrupted during a difficult chess game. In fact, it is even worse, because the interruption can influence the „game": A mistake made during the interruption, i.e. during the lower-level task, severely disturbs the higher-level task and may prevent the student from learning. This is exactly what leaded to the wrong solution x=18 in the above example: After deciding to subtract 3, the student should fully concentrate on subtracting 3 from both sides of the equation while „forgetting" the reason for choosing this equivalence transformation. But, in reality, the student starts the next line with „x=" simply *„because the transformation –3 was chosen in order to generate x= on the left hand side"*. But then, at the higher level, the student has the (wrong) impression that –3 simplified the equation as desired.

This continuous change of levels inevitably occurs in almost all topics in school mathematics. It appears to be one of the central problems in mathematics education that students have to learn a new ability/skill while still practising an „old" one.

Using a CAS (we use Derive) the learning process could be conducted as follows. First we enter the equation.

`5x−6=2x+15` (ENTER)

```
#1:   5·x − 6 = 2·x + 15
```

Then follows the input of the equivalence transformation (#1 is a reference to the above equation).

`#1−2x` (ENTER) , then simplify by clicking the button =

```
#2:   (5·x − 6 = 2·x + 15) − 2·x
#3:                                     3·x − 6 = 15
```

The simplification, i.e. the application of the equivalence transformation to both sides of the equation, was performed by the CAS. Then the student chooses the next equivalence transformation:

`#3+6` (ENTER) , then simplify with $\boxed{=}$

```
#4:   (3·x − 6 = 15) + 6
#5:                                    3·x = 21
```

We mimic a student who makes the above discussed mistake:

`#5−3` (ENTER) , then simplify with $\boxed{=}$

```
#6:   (3·x = 21) − 3
#7:                                  3·x − 3 = 18
```

It goes without saying that the CAS simplifies properly, hence the student receives an <u>immediate</u> feed-back that the transformation –3 was not successful (i.e. did not simplify the equation to the expected „*x*=“).

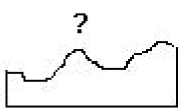
With this example we also meet again two issues that we discussed earlier: (1) The student *experiments* with possible equivalence transformations, hence this example also includes the aspect of experimental learning, and (2) the *immediacy* of the result of applying the transformation tallies with what we asked for in the section on visualization.

A student who follows the above CAS-supported exercise can fully concentrate on the (higher-level) skill of choosing an equivalence transformation. The lower-level skill of simplification is performed (at least for the moment) by the CAS.

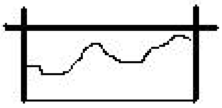
In [Kutzler 1998a,b] I give detailed instructions how to use Derive or the TI-92 (TI-89) to teach/learn solving of linear equations. The above algebraic approach is discussed in further detail as well as numeric and graphic approaches.

(3) The Scaffolding Method

In the above subsection on concentration we compared teaching mathematics with building a house. In the language of this metaphor the above mentioned problem of mathematics education translates into the problem of building a new storey on top of an incomplete storey. For example, as soon as we start building the storey of „choosing equivalence transformations“, the storey of „simplifying“ is still incomplete for many of our students. In mathematics teaching at school we simply don’t have enough time for waiting until all students have completed all previous storeys. The curriculum forces the teacher to continue with the next topic, independent of the progress of individual students. So, it remains to ask how a student can build a storey on top of an incomplete one.



The above example demonstrates how I suggest to answer this question: While the student learns the higher-level skill, the calculator solves all subproblems that require the lower-level skill. Using the language of the metaphor, the calculator is a scaffolding above the incomplete storey.



Using the example of solving a linear equation we demonstrated the use of Derive as a scaffolding above the simplification storey. In the sequel we apply the scaffolding method to another example, namely the solving of a system of linear equations with Gaussian elimination.

We use the system $2x+3y=4$, $3x-4y=5$. First, the student enters the equations:

`2x+3y = 4` (ENTER)
`3x-4y = 5` (ENTER)

```
#1:   2·x + 3·y = 4
#2:   3·x - 4·y = 5
```

Gaussian elimination requires us to choose a linear combination of the two equations such that one variable is eliminated. This is what needs to be learned. Everything else (simplification, substitution, solving of an equation in one variable) are prerequisites of which the teacher expects that the students are well enough trained.

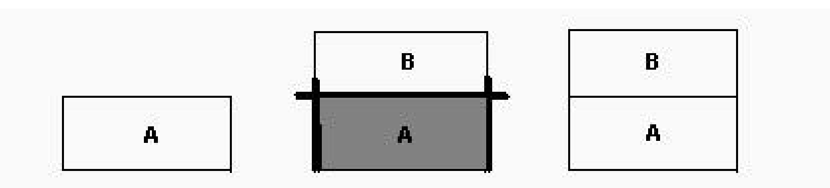
We try to eliminate y by adding four times the first equation and three times the second equation.

`4*#1+3*#2` (ENTER) then simplify with $\boxed{=}$

```
#3:   4·(2·x + 3·y = 4) + 3·(3·x - 4·y = 5)
#4:                                          17·x = 31
```

Voila. Variable y disappeared as requested. What is the practice of teaching this topic at school in a paper&pencil environment? Some students choose the right linear combination, but, due to a calculation error, the variable does not disappear. Other students choose a wrong linear combination, but the variable disappears because it „must“ disappear. For both groups of students their weakness in algebraic simplification is a stumbling block for successfully learning the basic technique of Gaussian elimination. Exactly those students lag behind more and more the „higher“ they get in the house of mathematics.

In the above exercise, the algebraic calculator is a scaffolding that compensates any weakness with the lower-level skills, hence it helps avoiding mistakes. In case the final teaching goal is to have students be able to solve systems of equations (or perform any other skill B) manually (because, for example, the students are assessed centrally at the end of the school year), then it is recommended to perform the following three steps. The first step is teaching and practising skill A. The second step is teaching and practising skill B, while using the algebraic calculator to solve all those subproblems that require skill A (i.e. the student can fully concentrate on learning skill B.) The third step is to combine skills A and B with no support from technology.



This is one of many conceivable methods of using CAS as pedagogical tools.

The temporary use of technology can help to break down the learning process into smaller, easier „digestible“ pieces. For less gifted students, who could not swallow the „big pieces“ we offered them in traditional mathematics teaching, this may be the only way of mastering these learning steps. They find it easier to keep track of the steps without getting lost (or screwed up) in details such as simplification. When comparing the teaching of mathematics with building a house, the use of technology compares with using a scaffolding.

The ***scaffolding method*** is any pedagogically justified sequence of using and not using technology for trivia lization, experimentation, visualization, or concentration either in the sense of automation or compensation.

The presented use of technology as a pedagogical tool is completely independent whether or not technology may be used during an exam. The scaffolding method aims at supporting the learning process, i.e. it can help reaching (traditional) teaching goals. Here, technology is only a training instrument. Like a home trainer can help us acquiring physical skills, the proper use of CAS can be a mathematical home training which helps acquiring intellectual/mathematical skills. Consequently, technology can and should be introduced as a pedagogical tool independent of changes to the curriculum or the assessment scheme. CAS can help at all levels of mathematics in secondary education.

In [Kutzler 1998c,d] I describe how one can use Derive or the TI-92 (TI-89) to treat the topic „solving of systems of linear equations“ at school. Besides giving further details on the above approach, the booklet also describes numeric and graphic methods as well as the substitution method.

(4) Concluding remarks

In Austria in 1991 all general high schools (Gymnasien) and technical high schools (Höhere Technische Lehranstalten) were equipped with the Derive computer algebra system. In the sequel a research project was conducted which became known as the „Austrian Derive Project“. The project involved 800 students who were taught regular mathematics with Derive. The results were published in the German language book [Heugl/Klinger/Lechner 1996], part of the results can also be found in the English language publication [Aspetsberger/Fuchs 1996]. In the academic year 1997/98 a bigger team carried out the „Austrian TI-92 Project I“ with 2,000 students using the TI-92 in regular mathematics classes, which was followed by the “Austrian TI-92 Project II” with 3,000 students. The results can be found in the internet at the address http://www.acdca.ac.at.

The Austrian and other investigations showed the following: If technology is used properly, it leads to

- more efficient teaching and learning,
- more inpedendent productive student activity,
- more student creativity,
- an increased importance of the teacher.

The teacher has the duty to accompany and direct the students at their partly individual voyage of discovery through the world of mathematics. Consequently the key to the success in teaching mathematics is a good teacher which is obtained through good teacher training. Not technology changes teaching, but technology can be a catalyst for teachers to change their teaching methods and teaching focus, aiming at a better teaching of mathematics.

In case you have questions or suggestions, please write to me at b.kutzler@eunet.at. A regularly updated collection of information about technology in mathematics education can be found at www.kutzler.com.

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Literatur

K Aspetsberger, K Fuchs, 1996 (guest editors): The International DERIVE Journal Vol 3, No. 1. (The whole issue was dedicated to the Austrian DERIVE project.) Hemel Hempstead:Research Information Ltd., 106 pages, ISSN 1351-0789.

B Buchberger, 1989: *Why Should Students Learn Integration Rules?* RISC-Linz Technical Report no. 89-7.0, Univ of Linz, Austria.

F Demana, B Waits, 1990: *The Role of Technology in Teaching Mathematics.* In: „The Mathematics Teacher“, National Council of Teachers of Mathematics, vol. 82, no. 1.

F Demana, B Waits, 1992: *A Computer for All Students.* In: „The Mathematics Teacher“, National Council of Teachers of Mathematics, vol. 84, no. 2.

F Demana, B Waits, 1994: *Graphing Calculator Intensive Calculus: A First Step in Calculus Reform for All Students*. In: „Proc. Of the Preparing for New Calculus Conference“ (ed. A. Solow), Mathematical Association of America.

A Einstein, 1956: *Out of My Later Years.* Carol Publ. Comp.

H Freudenthal, 1979: *Mathematik als pädagogische Aufgabe*. Klett Studienbücher.

W Herget, H Heugl, B Kutzler, E Lehmann, 2000: *Indispensable Manual Calculation Skills in a CAS Environment*. In: 'Ohio Journal of School Mathematics', Autumn 2000, no 42, pp. 13-20; also in: 'Exam Questions and Basic Skills in Technology-Supported Mathematics Teaching (Proceedings Portoroz'2000)', eds. V Kokol-Voljc etal, 6th ACDCA Summer Academy, Jul 2-5, 2000, Portoroz, Slovenia, pp. 13-26; also in: 'Micromath', vol. 16 (2000), no. 3, pp. 8-12.

H Heugl, W Klinger, J Lechner, 1996: *Mathematikunterricht mit Computeralgebra-Systemen (Ein didaktisches Lehrerbuch mit Erfahrungen aus dem österreichischen DERIVE-Projekt)*. Bonn:Addison-Wesley, 307 pages, ISBN 3-8273-1082-2.

B Kutzler, 1995: *Improving Mathematics Teaching with DERIVE*. Bromley:Chartwell-Bratt, 185 pages, ISBN 0-86238-422-2.

B Kutzler, 1996: *Introduction to the TI-92 (Handheld Computer Algebra)*. Hagenberg:bk teachware, 184 pages, ISBN 3-901769-02-1.

B Kutzler, 1998a: *Solving Linear Equations with Derive (Experimental Learning / Visualization / Scaffolding Method)*. Hagenberg:bk teachware, 36 pages, ISBN 3-901769-09-9.

B Kutzler, 1998b: *Solving Linear Equations with the TI-92 (Experimental Learning / Visualization / Scaffolding Method)*. Hagenberg:bk teachware, 36 pages, ISBN 3-901769-03-X.

B Kutzler, 1998c: *Solving Systems of Equations with Derive (Experimental Learning / Visualization / Scaffolding Method)*. Hagenberg:bk teachware, 44 Seiten, ISBN 3-901769-10-2.

B Kutzler, 1998d: *Solving Systems of Equations with the TI-92 (Experimental Learning / Visualization / Scaffolding Method)*. Hagenberg:bk teachware, 44 Seiten, ISBN 3-901769-05-6.

B Kutzler, 2000: *Two-Tier Exams as a Way to Let Technology In*. In: 'Exam Questions and Basic Skills in Technology-Supported Mathematics Teaching (Proceedings Portoroz'2000)', eds. V Kokol-Voljc etal, 6th ACDCA Summer Academy, Jul 2-5, 2000, Portoroz, Slovenia, pp. 121-124, see.

B Kutzler, 2001: *What Math Should We Teach When We Teach Math With CAS*. Technical Report.

N Marin, A Benarroch, 1994: *A Comparative Study of Pagetian and Constructivist Work on Conceptions Science*. In: "Int. J. of Science Education", vol. 16, no. 1, pp. 1-15.

J Piaget, 1972: *Die Entwicklung des Erkennens I – Das mathematische Denken*. Stuttgart: Klett.