12.8 Logical Reasoning: Proof

What you should learn

GOAL 1 Use logical reasoning and proof to prove a statement is true.

GOAL 2 Prove that a statement is false.

Why you should learn it

▼ To help solve **real-life** problems, such as defending a client in court in **Example 4.**

REAL LIFE

GOAL 1 PROVING A STATEMENT IS TRUE

LOGICAL REASONING Mathematics is believed to have begun with practical "rules of thumb" that were developed to deal with real-life problems, such as tax records, the surveying of fields, inventories of storehouses, business transactions between merchants, and astronomical calculations. Then, about 2500 years ago, Greek geometers developed a different approach toward mathematics. Starting with a handful of properties that they believed to be true, these geometers insisted on logical reasoning as the basis for developing more elaborate mathematical tools. These more elaborate tools, which can be proved to be true, are called *theorems*.

The basic properties of mathematics that mathematicians accept without proof are called **postulates** or **axioms.** Many of the rules that were discussed in Chapter 2, such as the properties of addition, the properties of multiplication, and the distributive property, fall into this category. The following chart provides a summary of the rules that underlie algebra.

CONCEPT SUMMARY — **THE BASIC AXIOMS OF ALGEBRA**

Let a, b, and c be real numbers.

Axioms of Addition and Multiplication

CLOSURE:	$a + b$ is a real number.	ab is a real number.
COMMUTATIVE:	$a + b = b + a$	$ab = ba$
ASSOCIATIVE:	$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$
IDENTITY:	$a + 0 = a,\ 0 + a = a$	$a(1) = a,\ 1(a) = a$
INVERSE:	$a + (-a) = 0$	$a\left(\frac{1}{a}\right) = 1,\ a \neq 0$

Axiom Relating Addition and Multiplication

DISTRIBUTIVE:	$a(b + c) = ab + ac$	$(a + b)c = ac + bc$

Axioms of Equality

ADDITION:	If $a = b$, then $a + c = b + c$.
MULTIPLICATION:	If $a = b$, then $ac = bc$.
SUBSTITUTION:	If $a = b$, then a can be substituted for b.

Once a list of axioms has been accepted, you can add more rules, formulas, and properties to the list. Some of the new concepts are definitions. You don't have to prove a definition, but it must be consistent with previous definitions and axioms. For instance, the definition of *reciprocal* does not have to be proved.

Other new statements, called **theorems**, do have to be proved. For instance, basic axioms have been used to prove the theorem that for all real numbers b and c, $c(-b) = -cb$. Once a theorem is proved, it can be used as a reason in proofs of other theorems.

EXAMPLE 1 *Proving a Theorem*

STUDENT HELP

Study Tip
When you are proving a new theorem, every step must be justified by an axiom, a definition, given information, or a previously proved theorem.

Use the definition of subtraction, $a - b = a + (-b)$, to prove the following theorem: $c(a - b) = ca - cb$.

SOLUTION

$c(a - b) = c[a + (-b)]$	Definition of subtraction
$= ca + c(-b)$	Distributive property
$= ca + (-cb)$	Theorem stated above
$= ca - cb$	Definition of subtraction

. . . .

A **conjecture** is a statement that is thought to be true but is not yet proved. Often it is a statement based on observation.

EXAMPLE 2 *Goldbach's Conjecture*

Christian Goldbach (1690–1764) thought the following statement might be true. It is now referred to as *Goldbach's Conjecture.*

Every even integer, except 2, is equal to the sum of two prime numbers.

The following list shows that every even number between 4 and 30 is equal to the sum of two prime numbers. Does this list prove that *every* even number greater than 2 is equal to the sum of two prime numbers?

$4 = 2 + 2$	$6 = 3 + 3$	$8 = 3 + 5$	$10 = 3 + 7$
$12 = 5 + 7$	$14 = 3 + 11$	$16 = 3 + 13$	$18 = 5 + 13$
$20 = 3 + 17$	$22 = 3 + 19$	$24 = 5 + 19$	$26 = 3 + 23$
$28 = 5 + 23$	$30 = 7 + 23$		

SOLUTION

This list of examples *does not* prove the conjecture. No number of examples can prove that the rule is true for *every* even integer greater than 2. (At the time this book is being written, no one has been able to prove or disprove Goldbach's Conjecture.)

GOAL 2 PROVING A STATEMENT IS FALSE

Sometimes people propose new rules but later find that they *are not* consistent with the axioms, definitions, and theorems in the accepted list. To show that a conjecture is false, you need only one counterexample.

EXAMPLE 3 *Finding a Counterexample*

Assign values to a and b to show that the following conjecture is false.

$$a + (-b) = (-a) + b.$$

SOLUTION

You can choose any values of a and b. For instance, let $a = 1$ and let $b = 2$.

Evaluate the left side of the equation.

$a + (-b) = 1 + (-2)$ Substitute 1 for a and 2 for b.

$= -1$ Simplify.

Evaluate the right side of the equation.

$(-a) + b = (-1) + 2$ Substitute 1 for a and 2 for b.

$= 1$ Simplify.

Because $-1 \neq 1$, you have shown one case in which $a + (-b) = (-a) + b$ is false. The counterexample of $a = 1$ and $b = 2$ is sufficient to prove that $a + (-b) = (-a) + b$ is false. If the conjecture were true, it would have to work for *any* choices of a and b.

STUDENT HELP

Look Back
For help with counter-examples, see p. 66.

INDIRECT PROOF In this lesson you have used direct proofs to prove that statements are true, and you have used counterexamples to prove that statements are false.

Another type of proof is **indirect proof**. To prove a statement indirectly, assume that the statement is false. If this assumption leads to an impossibility, then you have proved that the original statement is true. An indirect proof is also called a *proof by contradiction*.

FOCUS ON CAREERS

LAWYERS
Lawyers represent people in criminal and civil trials by presenting evidence supporting their client's case. They also give advice on legal matters.

CAREER LINK
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EXAMPLE 4 *Use of Contradiction in Real Life*

LAWYERS You are a lawyer. You have been assigned to defend someone accused of violating a law uptown at 10:00 A.M. on March 22. You argue that if guilty, your client must have been there at that time. You have a video of your client being interviewed by a TV reporter at a political rally on the other side of town at the same time.

You argue that it would be impossible for your client to be uptown at 10:00 A.M. and on the other side of town at 10:00 A.M. Therefore your client cannot be guilty.

EXAMPLE 5 *Using an Indirect Proof*

Use an indirect proof to prove that $\sqrt{2}$ is an irrational number.

SOLUTION

Assume that $\sqrt{2}$ is *not* irrational. Then $\sqrt{2}$ is rational and can be written as the quotient of two integers a and b that have no common factors other than 1.

$\sqrt{2} = \frac{a}{b}$ — **Assume $\sqrt{2}$ is a rational number.**

$2 = \frac{a^2}{b^2}$ — **Square each side.**

$2b^2 = a^2$ — **Multiply each side by b^2.**

This implies that 2 is a factor of a^2. Therefore 2 is also a factor of a. (You will prove this in Exercise 25 on page 763.) So a can be written as $2c$.

$2b^2 = (2c)^2$ — **Substitute $2c$ for a.**

$2b^2 = 4c^2$ — **Simplify.**

$b^2 = 2c^2$ — **Divide each side by 2.**

This implies that 2 is a factor of b^2 and also a factor of b. So 2 is a factor of both a and b. But this is impossible because a and b have no common factors. Therefore it must be impossible that $\sqrt{2}$ is a rational number. So you can conclude that $\sqrt{2}$ must be an irrational number.

GUIDED PRACTICE

Vocabulary Check ✓

1. Explain the difference between an axiom and a theorem.

2. What is the first step of an indirect proof?

Concept Check ✓

In Exercises 3–11, state the basic axiom of algebra that is represented.

3. $y(1) = y$ **4.** $2x + 3 = 3 + 2x$ **5.** $5(x + y) = 5x + 5y$

6. $(4x)y = 4(xy)$ **7.** $y + 0 = y$ **8.** $x + (-x) = 0$

9. $x\left(\frac{1}{x}\right) = 1$ **10.** $cd = dc$ **11.** $(x + y) + z = x + (y + z)$

12. Is subtraction closed for the positive real numbers? That is, if a and b are positive real numbers, must $(a - b)$ be a positive real number? Explain your thinking.

Skill Check ✓

13. Copy and complete the proof of the statement: If $a = b$ and $c = d$, then $ac = bd$. Each variable represents any real number.

$a = b$ — **Given**

$ac = bc$ — **Multiplication axiom of equality**

$c = d$ — **Given**

$bc = bd$ — **?**

$ac = bd$ — **?**

Practice and Applications

Student Help

Extra Practice to help you master skills is on p. 808.

14. Supplying Reasons Copy and complete the proof of the statement: For all real numbers a and b, $(a + b) - b = a$.

$(a + b) - b = (a + b) + (-b)$	Definition of subtraction
$(a + b) - b = a + [b + (-b)]$	Associative property of addition
$(a + b) - b = a + 0$	?
$(a + b) - b = a$	?

Proving Theorems In Exercises 15–17, prove the theorem. (Use the basic axioms of algebra and the definition of subtraction given in Example 1.)

15. If a and b are real numbers, then $a - b = -b + a$.

16. If a, b, and c are real numbers, then $(a - b) - c = a - (b + c)$.

17. If a is any real number, then $-1(a) = -a$.

18. Proving a Conjecture A student proposes the following conjecture. *The sum of the first n odd integers is* n^2. She gives four examples: $1 = 1^2$, $1 + 3 = 4 = 2^2$, $1 + 3 + 5 = 9 = 3^2$, and $1 + 3 + 5 + 7 = 16 = 4^2$. Do the examples prove her conjecture? Explain. Do you think the conjecture is true?

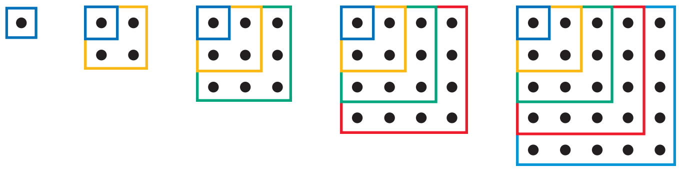
Finding a Counterexample In Exercises 19–21, find a counterexample to show that the statement is *not* true.

19. If a and b are real numbers, then $(a + b)^2 = a^2 + b^2$.

20. If a, b, and c are nonzero real numbers, then $(a \div b) \div c = a \div (b \div c)$.

21. If a and b are integers, then $a \div b$ is an integer.

22. Geometry Connection Explain how the diagrams below can be used to give a geometrical argument to support the conjecture in Exercise 18.



23. The Four-Color Problem A famous theorem states that any map can be colored with four different colors so that no two countries that share a border have the same color. No matter how the map shown at the right is colored with three different colors, at least two countries having a common border will have the same color. Does this map serve as a counterexample to the following proposal? Explain.

Any map can be colored with three different colors so that no two countries that share a border have the same color.

Student Help

Homework Help
Example 1: Exs. 14–17
Example 2: Ex. 18
Example 3: Exs. 19–21
Example 4: Ex. 24
Example 5: Exs. 25–27

INDIRECT PROOF In Exercises 24–27, use an indirect proof to prove that the conclusion is true.

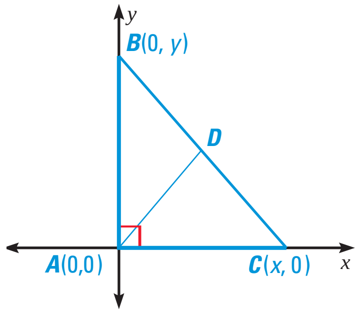
24. Your bus leaves a track meet at 4:30 P.M. and does not travel faster than 60 miles per hour. The meet is 45 miles from home. Your bus will not get you home in time for dinner at 5 P.M.

25. If p is an integer and p^2 is divisible by 2, then p is divisible by 2. (*Hint*: An odd number can be written as $2n + 1$, where n is an integer. An even number can be written as $2n$.)

26. If $a < b$, then $a + c < b + c$.

27. If $ac > bc$ and $c > 0$, then $a > b$.

28. PROOF USING THE MIDPOINT Let D represent the midpoint between B and C, as shown at the right. Prove that for any right triangle, the midpoint of its hypotenuse is equidistant from the three vertices of the triangle. In order to prove this, you must first find the distance, BC, between B and C. Using the distance formula, you get $BC = \sqrt{x^2 + y^2}$, so BD and CD must be $\frac{1}{2}\sqrt{x^2 + y^2}$.

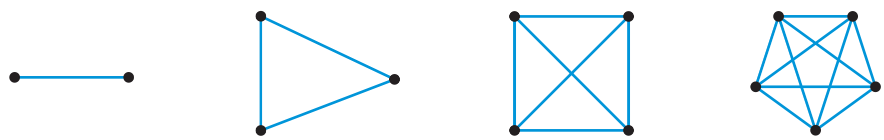


To help you with your proof, use the distance formula to find AD.

Test Preparation

29. MULTI-STEP PROBLEM In graph theory, a *complete graph* is one in which every pair of vertices is connected by part of the graph called an *edge*. The following graphs are complete.

Two vertices | Three vertices | Four vertices | Five vertices



a. Copy and complete the table.

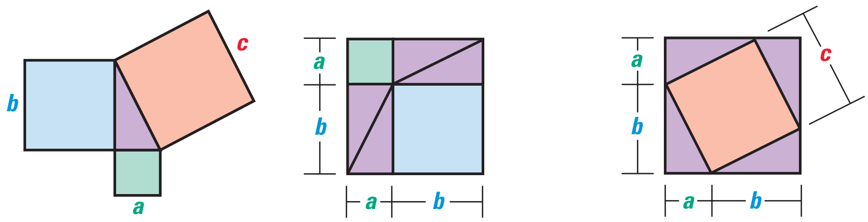
Vertices	2	3	4	5	6
Edges	1	3	?	?	?

b. Make a conjecture about the relationship between the number of vertices and the number of edges in a complete graph.

c. Use your conjecture to predict how many edges a complete graph with 10 vertices would have.

★ Challenge

30. PYTHAGOREAN THEOREM Explain how the following diagrams could be used to give a geometrical proof of the Pythagorean theorem.



EXTRA CHALLENGE
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MIXED REVIEW

FINDING SOLUTIONS **Decide how many solutions the equation has.** (Review 9.6)

31. $x^2 - 2x + 4 = 0$ **32.** $-2x^2 + 4x - 2 = 0$ **33.** $8x^2 - 8x + 2 = 0$

34. $x^2 - 14x + 49 = 0$ **35.** $-3x^2 - 5x + 1 = 0$ **36.** $6x^2 - x + 5 = 0$

37. $x^2 - 2x - 15 = 0$ **38.** $x^2 + 16x + 64 = 0$ **39.** $x^2 + 11x + 30 = 0$

FINDING SOLUTIONS **Decide whether the ordered pair is a solution of the inequality.** (Review 9.7)

40. $y < x^2 - 2x - 5$; $(1, 1)$ **41.** $y \geq 2x^2 - 8x + 8$; $(3, -2)$

42. $y \leq 2x^2 - 3x + 10$; $(-2, 20)$ **43.** $y > 4x^2 - 48x + 61$; $(1, 17)$

44. $y \geq x^2 + 4x$; $(-2, -4)$ **45.** $y < 3x^2 - 2x$; $(5, 10)$

46. $y > 3x^2 + 50x + 500$; $(-6, 100)$ **47.** $y \geq -x^2 + 3x - \frac{15}{4}$; $(2, -3)$

PERCENTS **Solve the percent problem.** (Review 11.2)

48. How much is 15% of \$15? **49.** 41 inches is what percent of 50 inches?

50. 100 is 1% of what number? **51.** 6 inches is what percent of 3 inches?

52. 1240 is 80% of what number? **53.** \$5 is 33% of what amount of money?

QUIZ 3

Self-Test for Lessons 12.6–12.8

Find the distance between the two points. Round the result to the nearest hundredth if necessary. Then find the midpoint between the two points. (Lesson 12.6)

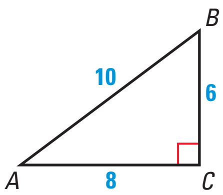
1. $(1, 3), (7, -9)$ **2.** $(-2, -5), (6, -11)$ **3.** $(0, 0), (8, -14)$

4. $(-8, -8), (-8, 8)$ **5.** $(3, 4), (-3, 4)$ **6.** $(1, 7), (-4, -2)$

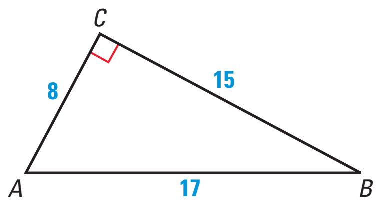
7. $(2, 0), (-2, -3)$ **8.** $(-3, 3), (4, 1)$ **9.** $(3, 4), (2, -4)$

Find the sine, the cosine, and the tangent of angles *A* and *B*. (Lesson 12.7)

10.



11.



12. Prove the theorem $(a - b)c = ac - bc$. Use only the basic axioms of algebra, the definition of subtraction, and the theorem $(-b)c = -bc$. (Lesson 12.8)

Find a counterexample to show that the statement is *not* true. (Lesson 12.8)

13. If a, b, and c are real numbers and $a < b$, then $ac < bc$.

14. If a and b are real numbers, then $-(a + b) = (-a) - (-b)$.