11.8 Rational Equations and Functions

What *you should learn*

GOAL 1 Solve rational equations.

GOAL 2 Graph rational functions.

Why *you should learn it*

▼ To model **real-life** problems as in the fundraising problem in **Exs. 53–54**.

REAL LIFE

GOAL 1 SOLVING RATIONAL EQUATIONS

A **rational equation** is an equation that contains rational expressions. Examples 1 and 2 show the two basic strategies for solving a rational equation.

EXAMPLE 1 *Cross Multiplying*

Solve $\frac{5}{y+2} = \frac{y}{3}$.

SOLUTION

$\frac{5}{y+2} = \frac{y}{3}$	Write original equation.
$5(3) = y(y+2)$	Cross multiply.
$15 = y^2 + 2y$	Simplify.
$0 = y^2 + 2y - 15$	Write in standard form.
$0 = (y+5)(y-3)$	Factor right side.

▶ If you set each factor equal to 0, you see that the solutions are -5 and 3. Check both solutions in the original equation.

Cross multiplying can be used only for equations in which each side is a single fraction. The second method, multiplying by the LCD, works for any rational equation. Multiplying by the LCD always leads to an equation with no fractions.

EXAMPLE 2 *Multiplying by the LCD*

Solve $\frac{2}{x} + \frac{1}{3} = \frac{4}{x}$.

SOLUTION

$\frac{2}{x} + \frac{1}{3} = \frac{4}{x}$	The LCD is $3x$.
$3x \cdot \frac{2}{x} + 3x \cdot \frac{1}{3} = 3x \cdot \frac{4}{x}$	Multiply each side by $3x$.
$6 + x = 12$	Simplify.
$x = 6$	Subtract 6 from each side.

✓CHECK

$\frac{2}{6} + \frac{1}{3} \stackrel{?}{=} \frac{4}{6}$	Substitute 6 for x.
$\frac{4}{6} = \frac{4}{6}$	Simplify.

EXAMPLE 3 *Factoring to Find the LCD*

STUDENT HELP

Study Tip
When you solve rational equations, be sure to check for extraneous solutions. Remember, values of the variable that make any denominator equal to 0 are excluded.

Solve $\frac{-4}{y-3} + 1 = \frac{-10}{y^2 + y - 12}$.

SOLUTION The denominator $y^2 + y - 12$ factors as $(y + 4)(y - 3)$, so the LCD is $(y + 4)(y - 3)$. Multiply each side of the equation by $(y + 4)(y - 3)$.

$$\frac{-4}{y-3} \cdot (y+4)(y-3) + 1 \cdot (y+4)(y-3) = \frac{-10}{y^2+y-12} \cdot (y+4)(y-3)$$

$$\frac{-4(y+4)\cancel{(y-3)}}{\cancel{y-3}} + (y+4)(y-3) = \frac{-10\cancel{(y+4)(y-3)}}{\cancel{(y+4)(y-3)}}$$

$$-4(y+4) + (y^2 + y - 12) = -10$$

$$-4y - 16 + y^2 + y - 12 = -10$$

$$y^2 - 3y - 28 = -10$$

$$y^2 - 3y - 18 = 0$$

$$(y - 6)(y + 3) = 0$$

The solutions are 6 and -3. Check both solutions in the original equation.

EXAMPLE 4 *Writing and Using a Rational Equation*

BATTING AVERAGE You have 35 hits in 140 times at bat. Your batting average is $\frac{35}{140} = 0.250$. How many consecutive hits must you get to increase your batting average to 0.300?

SOLUTION If your hits are consecutive, then you must get a hit each time you are at bat, so "future hits" is equal to "future times at bat."

VERBAL MODEL

$$\text{Batting average} = \frac{\text{Past hits} + \textbf{Future hits}}{\text{Past times at bat} + \textbf{Future times at bat}}$$

LABELS

Batting average	$= 0.300$	(no units)
Past hits	$= 35$	(no units)
Future hits	$= x$	(no units)
Past times at bat	$= 140$	(no units)
Future times at bat	$= x$	(no units)

ALGEBRAIC MODEL

$0.300 = \frac{35 + x}{140 + x}$	**Write algebraic model.**
$0.300(140 + x) = 35 + x$	**Multiply by LCD.**
$42 + 0.3x = 35 + x$	**Use distributive property.**
$7 = 0.7x$	**Subtract 0.3*x* and 35 from each side.**
$10 = x$	**Divide each side by 0.7.**

You need to get a hit in each of your next 10 times at bat.

FOCUS ON CAREERS

REAL LIFE
SPORTS REPORTER
Sports reporters gather statistics and prepare stories that cover all aspects of sports from local sporting events to international sporting events.

INTERNET CAREER LINK
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GOAL 2 Graphing Rational Functions

The inverse variation models you graphed in Lesson 11.3 are a type of *rational function*. A **rational function** is a function of the form

$$f(x) = \frac{\text{polynomial}}{\text{polynomial}}.$$

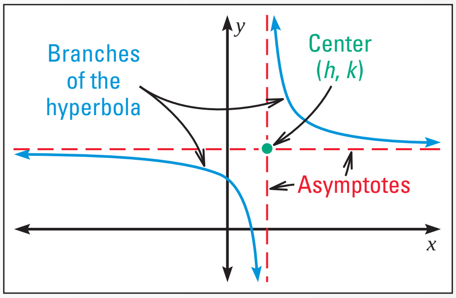
In this lesson you will learn to graph rational functions whose numerators and denominators are first-degree polynomials. Using long division, such a function can be written in the form

$$y = \frac{a}{x-h} + k.$$

RATIONAL FUNCTIONS WHOSE GRAPHS ARE HYPERBOLAS

The graph of the rational function $y = \frac{a}{x-h} + k$ is a **hyperbola** whose **center** is (h, k).

The vertical and horizontal lines through the center are the *asymptotes* of the hyperbola. An **asymptote** is a line that the graph approaches. While the distance between the graph and the line approaches zero, the asymptote is not part of the graph.

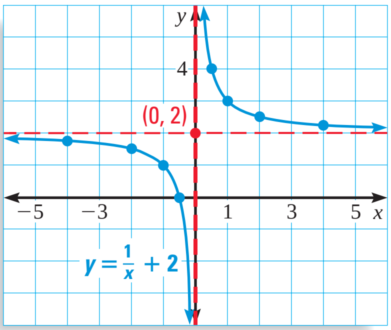


EXAMPLE 5 *Graphing a Rational Function*

Sketch the graph of $y = \frac{1}{x} + 2$.

SOLUTION Think of the function as $y = \frac{1}{x - \mathbf{0}} + \mathbf{2}$. You can see that the center is $(\mathbf{0}, \mathbf{2})$. The asymptotes can be drawn as dashed lines through the center. Make a table of values. Then plot the points and connect them with two smooth branches.

x	y
−4	1.75
−2	1.5
−1	1
−0.5	0
0	undefined
0.5	4
1	3
2	2.5
4	2.25



If you have drawn your graph correctly, it should be symmetric about the center (h, k). For example, the points $(0.5, 4)$ and $(-0.5, 0)$ are the same distance from the center $(0, 2)$, but in opposite directions.

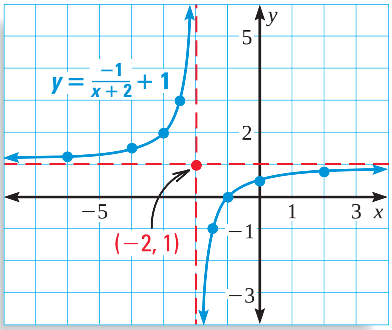
EXAMPLE 6 Graphing a Rational Function When h is Negative

Sketch the graph of $y = \frac{-1}{x+2} + 1$. Describe the domain.

SOLUTION

For this equation, $x - h = x + 2$, so $h = -2$. The graph is a hyperbola with center at $(-2, 1)$. Plot this point and draw a horizontal and a vertical asymptote through it. Then make a table of values, plot the points, and connect them with two smooth branches.

x	y
−6	1.25
−4	1.5
−3	2
−2.5	3
−2	undefined
−1.5	−1
−1	0
0	0.5
2	0.75



The domain is all real numbers except $x = -2$.

EXAMPLE 7 Rewriting before Graphing

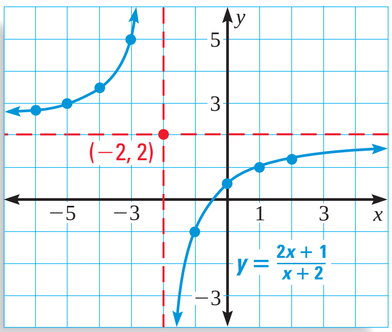
Sketch the graph of $y = \frac{2x+1}{x+2}$.

SOLUTION Begin by using long division to rewrite the rational function.

$$\begin{array}{r} 2 \\ x+2 \overline{)\,2x+1} \\ \underline{2x+4} \\ -3 \end{array}$$

$$y = \underbrace{2}_{\text{Quotient}} + \frac{\overbrace{-3}^{\text{Remainder}}}{x+2}, \text{ or } y = \frac{-3}{x+2} + 2$$

The graph is a hyperbola with center at $(-2, 2)$. Plot this point and draw a horizontal and a vertical asymptote through it. Make a table of values, plot the points, and connect them with two smooth branches.



When making a table of values, include values of x that are less than h and values of x that are greater than h.

Guided Practice

Vocabulary Check ✓

1. Describe the shape of a hyperbola. What is an asymptote of a hyperbola?

Concept Check ✓

In Exercises 2–4, find the least common denominator.

2. $\frac{1}{x}, \frac{x}{3}, \frac{2}{3x}$

3. $\frac{3}{4x}, \frac{1}{6x^2}, \frac{1}{8x^2}$

4. $\frac{1}{x^2 + 6x + 9}, \frac{1}{x + 3}$

5. Which of the equations in Exercises 6–11 can you solve using cross multiplying? Explain your answer.

Skill Check ✓

Solve the equation. Remember to check for extraneous solutions.

6. $\frac{3}{x} = \frac{x}{12}$

7. $\frac{-4x}{x + 1} = \frac{2}{x - 2}$

8. $\frac{x}{6} + \frac{15}{x} = \frac{5}{6}$

9. $\frac{4}{x^2 - 2x} = \frac{4}{3x - 6}$

10. $\frac{4}{x(x + 1)} = \frac{3}{x}$

11. $\frac{3}{x + 4} + \frac{4}{x} = \frac{-5}{x^2 + 4x}$

Find the center of the hyperbola. Draw the asymptotes and sketch the graph.

12. $y = \frac{4}{x + 5} - 3$

13. $y = \frac{2}{x - 2} + 3$

Practice and Applications

Student Help

Extra Practice
to help you master skills is on p. 807.

Cross Multiplying **Solve the equation by cross multiplying.**

14. $\frac{x}{5} = \frac{7}{3}$

15. $\frac{x}{10} = \frac{14}{5}$

16. $\frac{4}{x} = \frac{12}{5(x + 2)}$

17. $\frac{5}{x + 4} = \frac{5}{3(x + 1)}$

18. $\frac{6}{x + 2} = \frac{x}{4}$

19. $\frac{7}{x + 1} = \frac{5}{x - 3}$

Multiplying by the LCD **Solve the equation by multiplying each side by the least common denominator.**

20. $\frac{56}{x} = \frac{9 - x}{2}$

21. $\frac{x}{x + 9} = \frac{9}{x + 9} + 2$

22. $\frac{7}{3x - 12} - \frac{1}{x - 4} = \frac{2}{3}$

23. $\frac{1}{x - 4} + \frac{1}{x + 4} = \frac{22}{x^2 - 16}$

24. $\frac{1}{x - 4} + 1 = -\frac{7}{x^2 + x - 20}$

25. $2 + \frac{8}{x - 5} = \frac{x + 5}{x^2 - 25}$

Choosing a Method **Solve the equation.**

26. $\frac{1}{4} + \frac{4}{x} = \frac{1}{x}$

27. $\frac{-3x}{x + 1} = \frac{-2}{x - 1}$

28. $\frac{1}{5} - \frac{2}{5x} = \frac{1}{x}$

29. $\frac{x}{9} - \frac{8}{x} = \frac{1}{9}$

30. $\frac{x + 42}{x} = x$

31. $\frac{2}{x} - \frac{x}{8} = \frac{3}{4}$

32. $\frac{-3}{x + 7} = \frac{2}{x + 2}$

33. $\frac{2}{x + 3} + \frac{1}{x} = \frac{4}{3x}$

34. $\frac{10}{x + 3} - \frac{3}{5} = \frac{10x + 1}{3x + 9}$

35. $\frac{x + 3}{x - 5} = \frac{56 - 3x}{x^2 - 13x + 40}$

36. $\frac{8}{x + 4} + 1 = \frac{5x}{x^2 - 2x - 24}$

37. $\frac{x}{x - 11} - 1 = \frac{22}{x^2 - 5x - 66}$

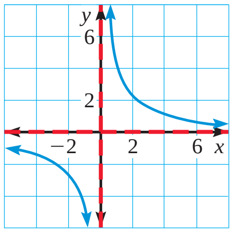
38. $\frac{2x}{x + 3} - \frac{x}{x + 7} = \frac{x^2 - 1}{x^2 + 10x + 21}$

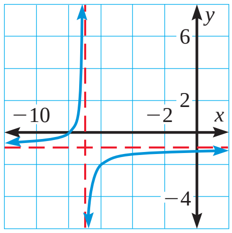
Student Help

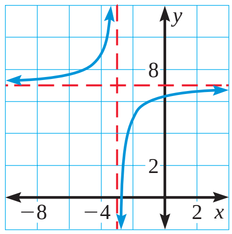
Homework Help

Example 1: Exs. 14–19, 26–38
Example 2: Exs. 20–38
Example 3: Exs. 20–38
Example 4: Exs. 51, 52
Example 5: Exs. 39–50
Example 6: Exs. 39–50
Example 7: Exs. 39–50

MATCHING GRAPHS **Match the function with its graph.**

A.


B.


C.


39. $y = -\dfrac{1}{x+7} - 1$ **40.** $y = \dfrac{-2}{x+3} + 7$ **41.** $y = \dfrac{4}{x}$

GRAPHING **In Exercises 42–50, graph the function. Describe the domain.**

42. $y = \dfrac{1}{x} + 4$ **43.** $y = \dfrac{1}{x-3} - 8$ **44.** $y = \dfrac{2}{x-4} + 6$

45. $y = -\dfrac{3}{x+1} + 8$ **46.** $y = \dfrac{6}{x+9} - 7$ **47.** $y = \dfrac{3x+11}{x+3}$

48. $y = \dfrac{-2x+11}{x-5}$ **49.** $y = \dfrac{9x-6}{x-1}$ **50.** $y = \dfrac{-5x+19}{x-3}$

51. **BATTING AVERAGE** After 50 times at bat, a major league baseball player has a batting average of 0.160. How many consecutive hits must the player get to raise his batting average to 0.250?

52. **TEST AVERAGES** You have taken 3 tests and have an average of 72 points. If you score 100 points on each of the rest of your tests, how many more tests do you need to take to raise your average to 88 points?

STUDENT HELP

INTERNET **HOMEWORK HELP** Visit our Web site www.mcdougallittell.com for help with Exs. 52–58.

FUNDRAISING **In Exercises 53 and 54, a library has received a single large contribution of \$5000. A walkathon is also being held in which each sponsor will contribute \$10. Let *x* represent the number of sponsors.**

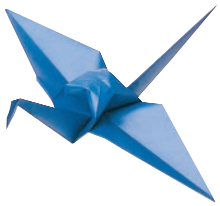
53. Write a function that represents the average contribution per person, including the single large contributor. Sketch the graph of the function.

54. Explain how such averages could be used to misrepresent actual contributions.

FOCUS ON APPLICATIONS

REAL LIFE **ORIGAMI CRANES** The art of paper folding has been practiced in Japan since the seventh century A.D. There is an ancient Japanese belief that folding 1000 cranes will make your wish come true.

ORIGAMI CRANES **In Exercises 55–58, you investigate how long it would take you and a friend to fold 1000 origami cranes. You take 2 minutes to fold 1 crane. Let *x* represent the number of minutes it might take a friend to fold 1 crane.**



55. Working alone, you would take 2000 minutes to fold 1000 cranes, so your work rate is $\dfrac{1}{2000}$ of the job per minute. Find your work rate per hour.

56. The expression $\dfrac{60}{1000x}$ is your friend's work rate per hour. Explain why.

57. Write an expression for the combined work rate per hour of you and your friend (the part of the job completed in 1 hour if you both work together).

58. Suppose that together you could complete 1000 cranes in 20 hours. Your combined work rate per hour would be $\dfrac{1}{20}$. To find the time your friend takes to fold 1 crane, set the expression in Exercise 57 equal to $\dfrac{1}{20}$ and solve for x.

INVERSE VARIATION AND RATIONAL FUNCTIONS **In Exercises 59 and 60, you will compare the types of graphs in 11.3 with those in this lesson.**

59. Graph $f(x) = \frac{6}{x}$ and $f(x) = \frac{6}{x-2} + 1$ in the same coordinate plane.

60. *Writing* How are the two graphs in Exercise 59 related? Do you think this relationship will hold for any set of functions $f(x) = \frac{a}{x-h} + k$ and $f(x) = \frac{a}{x}$? Explain.

61. **CHANGING CONSTANTS** Use a graphing calculator to graph $f(x) = \frac{a}{x}$ for at least ten values of a. Include positive, negative, and non-integer values. How does changing the value of a seem to affect the appearance of the graph?

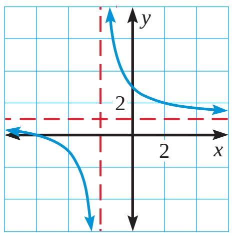
Test Preparation

62. **MULTIPLE CHOICE** What is the solution of the equation $\frac{9}{x+5} = \frac{7}{x-5}$?

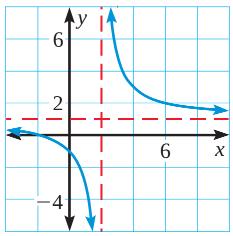
(A) 5 (B) 8 (C) 20 (D) 40 (E) 80

63. **MULTIPLE CHOICE** Choose the graph of the function $y = \frac{4}{x-2} + 1$.

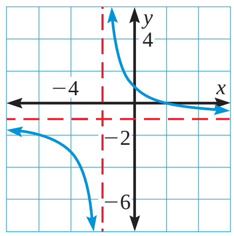
(A)



(B)



(C)



(D) None of the above

★ Challenge

RATIONAL FUNCTIONS **In Exercises 64–67, you will look for a pattern.**

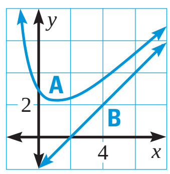
64. Copy and complete the table. Round your answers to the nearest hundredth.

x	0	20	40	60	80	100
$\frac{x^2+6}{x+2}$	?	?	?	?	?	?
$x-2$	?	?	?	?	?	?
$\frac{10}{x+2}$	?	?	?	?	?	?

65. What happens to the values of $\frac{x^2+6}{x+2}$, $(x-2)$, and $\frac{10}{x+2}$ as x increases?

66. Which is the graph of $y = x - 2$? of $y = \frac{x^2+6}{x+2}$?

67. **CRITICAL THINKING** Explain the relationship between the two graphs in Exercise 66. Use the table in Exercise 64 to help you.



Exs. 66 and 67

EXTRA CHALLENGE
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MIXED REVIEW

FUNCTION VALUES **Evaluate the function for $x = 0, 1, 2, 3,$ and 4.** **(Review 4.8 for 12.1)**

68. $f(x) = 4x$ **69.** $f(x) = -x + 9$ **70.** $f(x) = 3x + 1$

71. $f(x) = -x^2$ **72.** $f(x) = x^2 - 1$ **73.** $f(x) = \frac{x^2}{2}$

EVALUATING EXPRESSIONS **Evaluate the expression.** **(Review 8.1, 8.2)**

74. $2^4 \cdot 2^3$ **75.** $6^3 \cdot 6^{-1}$ **76.** $(3^3)^2$ **77.** $(-4^{-2})^{-1}$

RADICAL EXPRESSIONS **Simplify the radical expression.** **(Review 9.2 for 12.1)**

78. $\sqrt{50}$ **79.** $\sqrt{72}$ **80.** $\frac{1}{4}\sqrt{112}$ **81.** $\frac{1}{2}\sqrt{52}$

82. $\frac{1}{4}\sqrt{64}$ **83.** $\sqrt{256}$ **84.** $\frac{1}{5}\sqrt{625}$ **85.** $\sqrt{396}$

86. **ACCOUNT BALANCE** A principal of $500 is deposited in an account that pays 4% interest compounded yearly. Find the balance after 6 years. **(Review 8.5)**

QUIZ 3

Self-Test for Lessons 11.7 and 11.8

Divide. **(Lesson 11.7)**

1. $(x^2 - 8) \div (6x)$ **2.** $(6a^3 + 5a^2) \div (10a^2)$

3. $(x^2 + 16) \div (x + 4)$ **4.** $(2y^2 + 8y - 5) \div (2y - 5)$

5. $(8 + 4z + z^2) \div (3z - 6)$ **6.** $(12x^2 + 17x - 5) \div (3x + 2)$

Solve the equation. **(Lesson 11.8)**

7. $\frac{1}{2} + \frac{2}{t} = \frac{1}{t}$ **8.** $\frac{3}{x} = \frac{9}{2(x + 2)}$

9. $\frac{1}{x - 5} + \frac{1}{x + 5} = \frac{x + 3}{x^2 - 25}$ **10.** $\frac{7}{8} - \frac{16}{s - 2} = \frac{3}{4}$

In Exercises 11–13, sketch a graph of the function. **(Lesson 11.8)**

11. $f(x) = \frac{3}{x}$ **12.** $f(x) = \frac{6}{x - 4}$ **13.** $f(x) = \frac{3}{x + 1} - 2$

14. **TELEVISION STATIONS** The models are based on data collected by the Television Bureau of Advertising from 1988 to 1998 in the United States. Let t represent the number of years since 1988.

Number of UHF television stations: $U = 14t + 505$

Total number of commercial television stations: $T = 16t + 1048$

Use long division to find a model for the ratio of the number of UHF television stations to the total number of commercial television stations. **(Lesson 11.7)**