

# 12.4

## Completing the Square

### What you should learn

**GOAL 1** Solve a quadratic equation by completing the square.

**GOAL 2** Choose a method for solving a quadratic equation.

### Why you should learn it

▼ To solve **real-life** problems such as finding the path of the diver in **Ex. 71**.



### GOAL 1 SOLVING BY COMPLETING THE SQUARE

In the activity on page 729, you completed the square for expressions of the form  $x^2 + bx$  where  $b = 2, 4, 6,$  and  $8$ . In each case,  $x^2 + bx + \left(\frac{b}{2} \cdot \frac{b}{2}\right)$  was modeled by a square with sides of length  $x + \frac{b}{2}$ . By using FOIL to expand  $\left(x + \frac{b}{2}\right)\left(x + \frac{b}{2}\right)$ , you can show that this pattern holds for any real number  $b$ .

#### COMPLETING THE SQUARE

To complete the square of the expression  $x^2 + bx$ , add the square of half the coefficient of  $x$ .

$$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$$

#### EXAMPLE 1 Completing the Square

What term should you add to  $x^2 - 8x$  so that the result is a perfect square?

##### SOLUTION

The coefficient of  $x$  is  $-8$ , so you should add  $\left(\frac{-8}{2}\right)^2$ , or  $16$ , to the expression.

$$x^2 - 8x + \left(\frac{-8}{2}\right)^2 = x^2 - 8x + 16 = (x - 4)^2$$

#### EXAMPLE 2 Solving a Quadratic Equation

Solve  $x^2 + 10x = 24$  by completing the square.

##### SOLUTION

$$x^2 + 10x = 24$$

$$x^2 + 10x + 5^2 = 24 + 5^2$$

$$(x + 5)^2 = 49$$

$$x + 5 = \pm 7$$

$$x = -5 \pm 7$$

$$x = 2 \text{ or } x = -12$$

Write original equation.

Add  $\left(\frac{10}{2}\right)^2$ , or  $5^2$ , to each side.

Write left side as perfect square.

Find square root of each side.

Subtract 5 from each side.

Simplify.

► The solutions are 2 and  $-12$ . Check these in the original equation to see that both are solutions.

#### STUDENT HELP

##### Study Tip

When completing the square to solve an equation, remember that you must always add the term  $\left(\frac{b}{2}\right)^2$  to *both* sides of the equation.

### EXAMPLE 3 Solving a Quadratic Equation

Solve  $x^2 - x - 3 = 0$  by completing the square.

#### SOLUTION

$$x^2 - x - 3 = 0$$

$$x^2 - x = 3$$

$$x^2 - x + \left(-\frac{1}{2}\right)^2 = 3 + \frac{1}{4}$$

$$\left(x - \frac{1}{2}\right)^2 = \frac{13}{4}$$

$$x - \frac{1}{2} = \pm \frac{\sqrt{13}}{2}$$

$$x = \frac{1}{2} \pm \frac{\sqrt{13}}{2}$$

Write original equation.

Add 3 to each side.

Add  $\left(-\frac{1}{2}\right)^2$ , or  $\frac{1}{4}$ , to each side.

Write left side as perfect square.

Find square root of each side.

Add  $\frac{1}{2}$  to each side.

► The solutions are  $\frac{1}{2} + \frac{\sqrt{13}}{2}$  and  $\frac{1}{2} - \frac{\sqrt{13}}{2}$ . Check these in the original equation.  
.....

If the leading coefficient of the quadratic is not 1, you should divide each side of the equation by this coefficient *before* completing the square.

### EXAMPLE 4 The Leading Coefficient is Not 1

Solve  $2x^2 - x - 2 = 0$  by completing the square.

#### SOLUTION

$$2x^2 - x - 2 = 0$$

$$2x^2 - x = 2$$

$$x^2 - \frac{1}{2}x = 1$$

$$x^2 - \frac{1}{2}x + \left(-\frac{1}{4}\right)^2 = 1 + \frac{1}{16}$$

$$\left(x - \frac{1}{4}\right)^2 = \frac{17}{16}$$

$$x - \frac{1}{4} = \pm \frac{\sqrt{17}}{4}$$

$$x = \frac{1}{4} \pm \frac{\sqrt{17}}{4}$$

Write original equation.

Add 2 to each side.

Divide each side by 2.

Add  $\left(-\frac{1}{4}\right)^2$ , or  $\frac{1}{16}$ , to each side.

Write left side as perfect square.

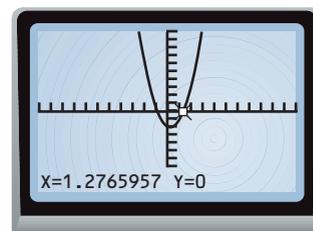
Find square root of each side.

Add  $\frac{1}{4}$  to each side.

► The solutions are  $\frac{1}{4} + \frac{\sqrt{17}}{4} \approx 1.28$  and

$$\frac{1}{4} - \frac{\sqrt{17}}{4} \approx -0.78.$$

✓ **CHECK** The graph generated by a graphing calculator appears to confirm the solutions.



#### STUDENT HELP

##### Study Tip

To solve an equation by completing the square, you first need to isolate the  $x^2$ -term and the  $x$ -term on one side of the equation.

#### STUDENT HELP



##### HOMEWORK HELP

Visit our Web site  
[www.mcdougallittell.com](http://www.mcdougallittell.com)  
for extra examples.

## GOAL 2 CHOOSING A SOLUTION METHOD

There are many connections among the five methods for solving a quadratic equation. For instance, in Example 4 you used completing the square to solve the equation, and you used a graphing approach to confirm that the solutions are reasonable.

In the following activity, you will use completing the square to develop the quadratic formula.

### ACTIVITY

Developing  
Concepts

### Investigating the Quadratic Formula

Consider a general quadratic equation

$$ax^2 + bx + c = 0 \quad \text{where } a \neq 0.$$

Perform the following steps. Then describe how your result is related to the quadratic formula.

- 1 Subtract  $c$  from each side of the equation  $ax^2 + bx + c = 0$ .
- 2 Divide each side by  $a$ .
- 3 Add the square of half the coefficient of  $x$  to each side.
- 4 Write the left side as a perfect square.
- 5 Use a common denominator to express the right side as a single fraction.
- 6 Find the square root of each side. Include  $\pm$  on the right side.
- 7 Solve for  $x$  by subtracting the same term from each side.
- 8 Use a common denominator to express the right side as a single fraction.

You have learned the following five methods for solving quadratic equations. The table will help you choose which method to use to solve an equation.

#### CONCEPT SUMMARY

#### METHODS FOR SOLVING $ax^2 + bx + c = 0$

Method	Lesson	Comments
FINDING SQUARE ROOTS	9.1	Efficient way to solve $ax^2 + c = 0$ .
GRAPHING	9.4	Can be used for <i>any</i> quadratic equation. May give only approximate solutions.
USING THE QUADRATIC FORMULA	9.5	Can be used for <i>any</i> quadratic equation. Always gives exact solutions to the equation.
FACTORING	10.5–10.8	Efficient way to solve a quadratic equation if the quadratic expression can be factored easily.
COMPLETING THE SQUARE	12.4	Can be used for <i>any</i> quadratic equation but is simplest to apply when $a = 1$ and $b$ is an even number.

### EXAMPLE 5 Choosing a Solution Method

Choose a method to solve the quadratic equation. Explain your choice.

a.  $3x^2 - 15 = 0$

b.  $2x^2 + 3x - 4 = 0$

#### SOLUTION

- a. Because this quadratic equation has the form  $ax^2 + c = 0$ , it is most efficiently solved by finding square roots.

$$3x^2 - 15 = 0$$

Write original equation.

$$3x^2 = 15$$

Add 15 to each side.

$$x^2 = 5$$

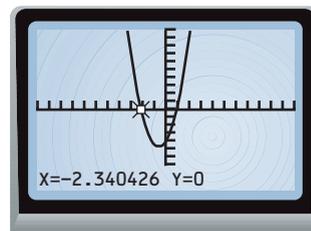
Divide each side by 3.

$$x = \pm\sqrt{5}$$

Find square roots.

- ▶ The solutions are  $\sqrt{5}$  and  $-\sqrt{5}$ .

- b. ▶ Because this quadratic equation is not easily factored, you can use a graphing calculator to approximate the solutions. **TRACE** to estimate the  $x$ -intercepts. The approximate solutions are  $-2.3$  and  $0.9$ .



- ✓ **CHECK** Using the quadratic formula, the exact solutions are

$$\frac{-3 + \sqrt{41}}{4} \text{ and } \frac{-3 - \sqrt{41}}{4}$$

### EXAMPLE 6 Choosing the Quadratic Formula

#### FOCUS ON APPLICATIONS



**REAL LIFE** **VETTISFOSS FALLS** is Norway's second-highest waterfall. There are many waterfalls in Norway. In fact, Norway generates more hydroelectric power per person than any other country.

**VETTISFOSS FALLS** The Vettisfoss waterfall falls over a vertical cliff. The path of the water can be modeled by  $h = -6.03x^2 + 901$  where  $h$  is the height (in feet) above the lower river and  $x$  is the horizontal distance (in feet) from the base of the cliff. How far from the base of the cliff does the water hit the lower river?

#### SOLUTION

When the falling water hits the lower river,  $h = 0$ . The quadratic equation  $0 = -6.03x^2 + 901$  cannot be factored easily and cannot be solved easily by completing the square. The quadratic formula is a good choice.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Write quadratic formula.

$$x = \frac{0 \pm \sqrt{0 - 4(6.03)901}}{2(-6.03)}$$

Substitute values for  $a$ ,  $b$ , and  $c$ .

$$\approx 12.3 \text{ or } -12.3$$

Use a calculator.

- ▶ Distance must be positive, so the negative solution is extraneous. The water hits the lower river about 12.3 feet from the base of the cliff. Check this solution in the original equation.

## GUIDED PRACTICE

**Vocabulary Check** ✓

1. The leading coefficient of the polynomial  $3x^2 - 8x + 4$  is  $\underline{\quad}$ .

**Concept Check** ✓

2. Explain why completing the square of the expression  $x^2 + bx$  is easier to do when  $b$  is an even number.

**Skill Check** ✓

**Find the term that should be added to the expression to create a perfect square trinomial.**

3.  $x^2 + 20x$

4.  $x^2 + 50x$

5.  $x^2 - 10x$

6.  $x^2 - 14x$

7.  $x^2 - 22x$

8.  $x^2 + 100x$

9. Solve  $x^2 - 3x = 8$  by completing the square. Solve the equation by using the quadratic formula. Which method did you find easier?

**Solve by completing the square.**

10.  $x^2 - 2x - 18 = 0$

11.  $x^2 + 14x + 13 = 0$

12.  $3x^2 + 4x - 1 = 0$

13.  $3x^2 - 7x + 6 = 0$

**Choose a method to solve the quadratic equation. What method did you use? Explain your choice.**

14.  $x^2 - x - 2 = 0$

15.  $3x^2 + 17x + 10 = 0$

16.  $x^2 - 9 = 0$

17.  $-3x^2 + 5x + 5 = 0$

18.  $x^2 + 2x - 14 = 0$

19.  $3x^2 - 2 = 0$

## PRACTICE AND APPLICATIONS

### STUDENT HELP

**Extra Practice**  
to help you master  
skills is on p. 808.

**PERFECT SQUARE TRINOMIALS** Find the term that should be added to the expression to create a perfect square trinomial.

20.  $x^2 - 12x$

21.  $x^2 + 8x$

22.  $x^2 + 21x$

23.  $x^2 - 22x$

24.  $x^2 + 11x$

25.  $x^2 - 40x$

26.  $x^2 + 0.4x$

27.  $x^2 + \frac{3}{4}x$

28.  $x^2 + \frac{4}{5}x$

29.  $x^2 - 5.2x$

30.  $x^2 - 0.3x$

31.  $x^2 + \frac{2}{3}x$

**COMPLETING THE SQUARE** Solve the equation by completing the square.

32.  $x^2 + 10x = 39$

33.  $x^2 + 16x = 17$

34.  $x^2 - 24x = -44$

35.  $x^2 - 8x + 12 = 0$

36.  $x^2 + 5x - \frac{11}{4} = 0$

37.  $x^2 + 11x + \frac{21}{4} = 0$

38.  $x^2 - \frac{2}{3}x - 3 = 0$

39.  $x^2 + \frac{3}{5}x - 1 = 0$

40.  $x^2 + x - 1 = 0$

41.  $4x^2 + 4x - 11 = 0$

42.  $3x^2 - 24x - 1 = 0$

43.  $4x^2 - 40x - 7 = 0$

44.  $2x^2 - 8x - 13 = 7$

45.  $5x^2 - 20x - 20 = 5$

46.  $3x^2 + 4x + 4 = 3$

47.  $4x^2 + 6x - 6 = 2$

48.  $6x^2 + 24x - 41 = 0$

49.  $20x^2 - 120x - 109 = 0$

### STUDENT HELP

#### HOMEWORK HELP

**Example 1:** Exs. 20–31  
**Example 2:** Exs. 32–40  
**Example 3:** Exs. 35–40  
**Example 4:** Exs. 41–49  
**Example 5:** Exs. 50–67  
**Example 6:** Ex. 71

**CHOOSING A METHOD** Choose a method to solve the quadratic equation. Explain your choice.

50.  $x^2 - 5x - 1 = 0$       51.  $4x^2 - 12 = 0$       52.  $n^2 + 5n - 24 = 0$   
 53.  $9a^2 - 25 = 0$       54.  $x^2 - x - 20 = 0$       55.  $x^2 + 6x - 55 = 0$

**SOLVING EQUATIONS** Solve the quadratic equation.

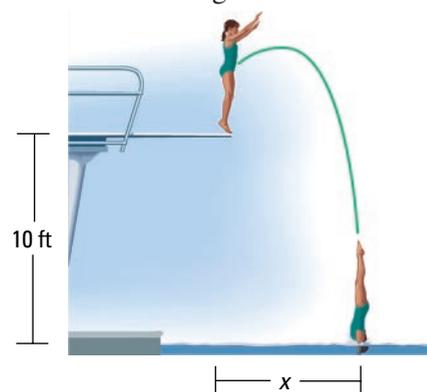
56.  $x^2 - 10x = 0$       57.  $c^2 + 2c - 26 = 0$       58.  $8x^2 + 14x = -5$   
 59.  $x^2 - 16 = 0$       60.  $x^2 + 12x + 20 = 0$       61.  $x^2 - 4x = \frac{5}{6}$   
 62.  $4x^2 + 4x + 1 = 0$       63.  $13x^2 - 26x = 0$       64.  $4p^2 - 12p + 5 = 0$   
 65.  $7z^2 - 46z = 21$       66.  $11x^2 - 22 = 0$       67.  $x^2 + 20x + 10 = 0$

**GEOMETRY CONNECTION** In Exercises 68–70, make a sketch and write a quadratic equation to model the situation. Then solve the equation.

68. In art class you are designing the floor plan of a house. The kitchen is supposed to have 150 square feet of space. What should the dimensions of the kitchen be if you want it to be square?  
 69. A rectangle is  $2x$  feet long and  $x + 5$  feet wide. The area is 600 square feet. What are the dimensions of the rectangle?  
 70. The height of a triangle is 4 more than twice its base. The area of the triangle is 60 square centimeters. What are the dimensions of the triangle?  
 71.  **DIVING** The path of a diver diving from a 10-foot high diving board is

$$h = -0.44x^2 + 2.61x + 10$$

where  $h$  is the height of the diver above water (in feet) and  $x$  is the horizontal distance (in feet) from the end of the board. How far from the end of the board will the diver enter the water?



**FOCUS ON APPLICATIONS**



 **PENGUINS** dive to capture food. Adelle penguins can stay underwater for seven minutes.

 **APPLICATION LINK**  
[www.mcdougallittell.com](http://www.mcdougallittell.com)

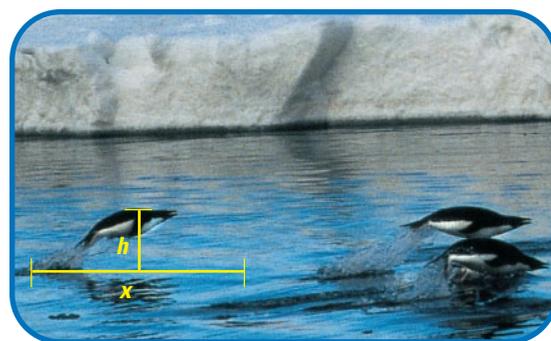
 **PENGUINS** In Exercises 72 and 73, use the following information.

You are on a research boat in the ocean. You see a penguin jump out of the water. The path followed by the penguin is given by

$$h = -0.05x^2 + 1.178x$$

where  $h$  is the height (in feet) the penguin jumps out of the water and  $x$  is the horizontal distance (in feet) traveled by the penguin over the water.

72. Sketch a graph of the equation.  
 73. How many horizontal feet did the penguin travel over the water before reaching its maximum height?



## Test Preparation



74. **MULTIPLE CHOICE** Which of the following is a solution of the equation  $2x^2 + 8x - 25 = 5$ ?
- (A)  $\sqrt{17} + 1$     (B)  $-\sqrt{19} - 2$     (C)  $\sqrt{17} - 2$     (D)  $\sqrt{21} - 2$
75. **MULTIPLE CHOICE** What term should you add to  $x^2 - \frac{1}{2}x$  so that the result is a perfect square trinomial?
- (A)  $\frac{1}{2}$     (B)  $\frac{1}{4}$     (C)  $\frac{1}{16}$     (D)  $\frac{1}{32}$
76. **MULTIPLE CHOICE** Solve  $x^2 + 8x - 2 = 0$ .
- (A)  $-4 \pm 3\sqrt{2}$     (B)  $4 \pm 3\sqrt{2}$     (C)  $-4 \pm 2\sqrt{2}$     (D)  $4 \pm \sqrt{16}$

## ★ Challenge

**VERTEX FORM** The *vertex form* of a quadratic function is  $y = a(x - h)^2 + k$ . Its graph is a parabola with vertex at  $(h, k)$ . In Exercises 77–79, use completing the square to write the quadratic function in vertex form. Then give the coordinates of the vertex of the graph of the function.

77.  $y = x^2 + 10x + 25$     78.  $y = 2x^2 + 12x + 13$     79.  $y = -x^2 - 5x + 6$

80. **QUADRATIC FORMULA** Explain why the quadratic formula gives solutions only if  $a \neq 0$  and  $b^2 - 4ac \geq 0$ .

### EXTRA CHALLENGE

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## MIXED REVIEW

**MEASURES OF CENTRAL TENDENCY** Find the mean, the median, and the mode of the collection of numbers. (Review 6.6)

81. 1, 5, 2, 4, 3, 6, 1    82. 9, 6, 10, 14, 10, 3  
83. -6, 20, -8, -18, 10    84. 17, 9, 11, 15, 4, 15, 8, 3, 11

**SOLVING LINEAR SYSTEMS** Solve the linear system. (Review 7.2, 7.3)

85.  $y = 4x$     86.  $3x + y = 12$     87.  $2x - y = 8$   
 $x + y = 10$      $9x - y = 36$      $2x + 2y = 2$

**SOLVING EQUATIONS** Solve the equation. (Review 9.1, 9.2 for 12.5)

88.  $16 + x^2 = 64$     89.  $x^2 + 81 = 144$     90.  $x^2 + 25 = 81$   
91.  $4x^2 - 144 = 0$     92.  $x^2 - 30 = -3$     93.  $x^2 = \frac{9}{25}$

**SKETCHING GRAPHS** Sketch the graph of the function. (Review 9.3)

94.  $y = x^2 + x + 2$     95.  $y = -3x^2 - x - 4$     96.  $y = 2x^2 - 3x + 4$

**SOLVING EQUATIONS** Solve the equation. (Review 10.4)

97.  $(x + 4)(x - 8) = 0$     98.  $(x - 3)(x - 2) = 0$     99.  $(x + 5)(x + 6) = 0$   
100.  $(x + 4)^2 = 0$     101.  $(x - 3)^2 = 0$     102.  $6(x - 14)^2 = 0$

**FACTORING TRINOMIALS** Factor the trinomial if possible. (Review 10.5, 10.6)

103.  $x^2 + x - 20$     104.  $x^2 - 10x + 24$     105.  $x^2 + 2x + 4$   
106.  $3x^2 - 15x + 18$     107.  $2x^2 - x - 3$     108.  $14x^2 - 19x - 3$